Extra Practice Problems 4

Here's a set of practice problems you can work through to help prepare for the upcoming midterm exam. We'll release solutions and another set of practice problems on Friday.

Problem One: Binary Relations

A binary relation R over a set A is called a ***Euclidean relation*** if the following is true about R:

$$\forall x \in A.\ \forall y \in A.\ \forall z \in A.\ (xRy \land xRz \to yRz)$$

The terminology comes from an axiom given by Euclid in his classical text *The Elements*: "things which equal the same thing also equal one another."

i. Let $k \in \mathbb{N}$ be a positive natural number. Is $\equiv_k$ Euclidean? How about $\leq$?

ii. Let R be an arbitrary binary relation over some set A. Prove that R is an equivalence relation if and only if it is reflexive and Euclidean.

Here's an unrelated question about relations.

iii. How many different equivalence relations are there over the set $\{a, b, c\}$? *(Hint: what structure are equivalence relations supposed to capture?)*

Problem Two: Cardinality

We have a good intuitive feel for what it means for a set to be finite or infinite – an infinite set contains infinitely many elements, and a finite set contains only finitely many. However, this definition is somewhat circular – if you don't already know what "infinite" and "finite" mean, these definitions won't really help you!

When set theory was first being developed, the mathematician Richard Dedekind came up with a proposed definition of what it means for a set to be infinite, which we now call ***Dedekind-infiniteness***. A set S is called ***Dedekind-infinite*** if there is at least one injection $f : S \to S$ that isn't a bijection. This question explores properties of Dedekind-infiniteness.

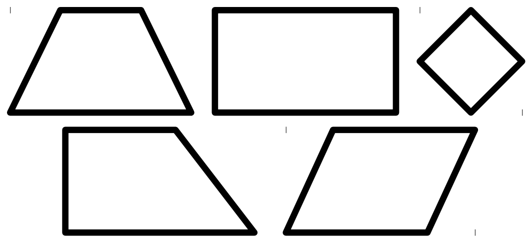
i. Prove or disprove: $\mathbb{N}$ is a Dedekind-infinite set.

ii. Prove or disprove: $[0, 1]$ is a Dedekind-infinite set. Here, $[0, 1]$ is the set of all real numbers between 0 and 1, inclusive.

iii. Prove that if S is a set and $|S| = k$ for some natural number k, then S is not Dedekind-infinite. *(Hint: proceed by contradiction and use the pigeonhole principle.)*

Problem Three: The Pigeonhole Principle

Suppose that you color every point in the real plane one of four colors (say, red, green, blue, and yellow). Prove that no matter how you color the plane, there will always be a trapezoid whose corners are all the same color. (Recall that a trapezoid is a quadrilateral with at least two parallel sides.) For example, all of the following figures are trapezoids:



(Hint: Try placing a specially-constructed object – say, a grid of dots – into the plane such that no matter how that object is colored, the object always contains a trapezoid whose corners are the same color.)

Problem Four: Regular Languages

For each of the following, show that the given language is regular by designing a DFA or NFA for it and by designing a regular expression for it.

i. Let $\Sigma = \{a, b\}$. Show that Σ^* is regular via a DFA/NFA and a regular expression.

ii. Let $\Sigma = \{a, b, c\}$. Let $L = \{ w \in \Sigma^* \mid$ any b's in w appear after the first c in $w \}$. Show that L is regular via a DFA/NFA and a regular expression.

iii. Let $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let $L = \{ w \in \Sigma^* \mid$ w is the base-10 representation of an even number and w has no extraneous leading zeros $\}$. Show that L is regular via a DFA/NFA and a regular expression.

Problem Five: Nonregular Languages

Let $\Sigma = \{a, b\}$ and let $L = \{ w \in \Sigma^* \mid w$ has odd length and its middle character is $a \}$. Prove that L is not a regular language.

Problem Six: Context-Free Languages

Let $\Sigma = \{$ **(**, **)** $\}$. Let $L = \{ w \in \Sigma^* \mid w$ is a prefix of a string of balanced parentheses $\}$. For example, since the string **((()()))** is a string of balanced parentheses, we see that $\varepsilon \in L$, **(** $\in L$, **((** $\in L$, **(((** $\in L$, **((()** $\in L$, **((()(** $\in L$, etc. Note that since any string is a prefix of itself, any string of balanced parenthees belongs to L. However, **(()))** $\notin L$, **((()))())** $\notin L$, and **)** $\notin L$.

Design a context-free grammar for L. *(Hint: Think about the intuition we used to get the initial grammar for balanced parentheses, then think about where the "cut" is made to form the prefix.)*